THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Tutorial 8

1. Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{Q}$ but discontinuous on $\mathbb{R}\backslash\mathbb{Q}$. (**Hints:** Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$. Use the continuity of f on $\mathbb{Q}$ and the density of $\mathbb{Q}$ to construct a nested sequence of closed bounded intervals I_n such that $r_n \notin I_{n+1}$ and that f is continuous on $\cap_{n=1}^{\infty} I_n$.)

2. If $f : [0,1] \to \mathbb{R}$ is continuous and has only rational (respectively,irrational) values, must f be a constant?

3. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a fixed point. ($c \in [0,1]$ is said to be fixed point of f is $f(c) = c$.)

4. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_\delta(x)$ of x. Prove that f is bounded on I. Can the closedness condition be dropped?

5. Determine if the following functions are uniformly continuous:

   (a) $f(x) : (0,1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$,

   (b) $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$,

   (c) $f : [0,M) \to \mathbb{R}$ defined by $f(x) = x^2$, where $M > 0$,

   (d) $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = x^2$,

   (e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^2+1}$,

   (f) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.